STABILITY OF COUETTE FLOW OF THE ANOMALOUS GRAD FLUID

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We investigate the stability of a steady flow of a Grad [1 and 2] fluid in the space between two rotating, infinitely long, concentric cylinders. We assume that the clearance between the cylinders is small, that they rotate both in the same direction and that small perturbations are axially symmetric [3]. We also assume that couple stresses are absent and that both, dimensionless relaxation time and dimensionless dynamic viscosity are small. We show that the critical Taylor number for a Grad fluid is larger than that for a Newtonian fluid. An approximate method of calculating the critical Taylor number is given, some properties of the spectrum of eigenvalues are noted and an analogy is drawn between the problem of stability of the flow of a Grad fluid, and of a viscoplastic fluid [5].

1. In the absence of couple stresses, the equations of motion for a viscous incompressible fluid have, in the cylindrical (r, θ, z) coordinate system, the form [1 and 2]

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} =$$

$$= -\frac{\partial p}{\partial r} + \frac{1+\delta}{R}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \frac{2\delta}{R}\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} +$$

$$+ \frac{1+\delta}{R}\left(\Delta v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \frac{2\delta}{R}\left(-\frac{\partial \omega_z}{\partial r} + \frac{\partial \omega_r}{\partial z}\right)$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1+\delta}{R}\Delta v_z + \frac{2\delta}{R}\left[\frac{1}{r}\frac{\partial (r\omega_\theta)}{\partial r} - \frac{1}{r}\frac{\partial \omega_r}{\partial \theta}\right]$$

$$\Delta = \frac{\partial^2(\ldots)}{\partial r^2} + \frac{1}{r}\frac{\partial(\ldots)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\ldots)}{\partial \theta^2} + \frac{\partial^2(\ldots)}{\partial z^2}$$

$$\frac{\partial \omega_r}{\partial t} + v_r \frac{\partial \omega_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_r}{\partial \theta} + v_z \frac{\partial \omega_r}{\partial z} - \frac{v_\theta \omega_\theta}{r} = \tau\omega_r + \frac{\tau}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) \qquad (1.1)$$

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0$$

$$\frac{\partial \omega_\theta}{\partial t} + v_r \frac{\partial \omega_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_\theta}{\partial \theta} + v_z \frac{\partial \omega_\theta}{\partial z} + \frac{v_\theta \omega_r}{r^2} = -\tau\omega_\theta + \frac{\tau}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)$$

$$\frac{\partial \omega_z}{\partial t} + v_r \frac{\partial \omega_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_z}{\partial \theta} + v_z \frac{\partial \omega_z}{\partial z} = -\tau\omega_z + \frac{\tau}{2}\left[\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$R = \frac{\Omega_1 R_1^2}{\nu}, \quad \tau = \frac{4\nu_r}{J\Omega_1}, \quad \delta = \frac{\nu_r}{\nu}$$

Here v_r, v_θ and v_z are the dimensionless projections of the velocity vector; ω_r, ω_θ and ω_z are the dimensionless projections of the mean angular velocity vector of fluid molecules; p is the dimensionless equilibrium pressure, ν is the kinematic Newtonian viscosity, ν_r is the kinematic radial viscosity, J is a scalar constant of the fluid with the dimension of the moment of inertia of the unit mass, Ω_1 and R_1 are the angular velocity and radius of the inner cylinder, respectively, τ is the dimensionless relaxation time and δ is the dimensionless radial viscosity which defines the measure of asymmetry of the stress tensor. Eqs. (1.1) have the exact solution

$$v_r^\circ = v_z^\circ = \omega_r^\circ = \omega_\theta^\circ = 0, \qquad v_\theta^\circ = ar + b/r = v_0, \qquad \omega_z^\circ = a$$
$$a = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{(R_2^2 - R_1^2)\,\Omega_1}, \qquad b = \frac{(\Omega_1 - \Omega_2)\,R_2^2}{\Omega_1 (R_2^2 - R_1^2)} \tag{1.2}$$

where Ω_2 and R_2 are the angular velocity and radius of the outer cylinder, respectively.

Solution (1.2) describes a steady flow of fluid in the space between rotating cylinders.

To see how this solution behaves under relatively small, axially symmetric perturbations, we shall investigate a unsteady solution of (1.1), which has the form [3]

$$v_r = v_r', \quad v_\theta = v_0 + v_\theta', \quad v_z = v_z', \quad p = p^\circ + p', \quad \omega_r = \omega_r'$$
$$\omega_\theta = \omega_\theta', \quad \omega_z = \omega_z^\circ + \omega_z', \quad v_r' = -\partial\psi'/\partial z, \; v_z' = r^{-1}\partial(r\psi')\partial r \tag{1.3}$$
$$v_\theta' = v(r)\,e^{\sigma t + i\lambda z}, \qquad \psi' = i\psi(r)\,e^{\sigma t + i\lambda z}$$
$$\omega_r' = \alpha(r)\,e^{\sigma t + i\lambda z}, \qquad \omega_\theta' = \beta(r)\,e^{\sigma t + i\lambda z}, \qquad \omega_z' = \gamma(r)\,e^{\sigma t + i\lambda z}$$

where λ is a real parameter and σ is complex.

Eqs. (1.1) and (1.3) yield [3] the following relations for ψ and v:

$$[L - \lambda^2 + (1+\delta)^{-1}\sigma R]\,(L - \lambda^2)\,\psi = 2\,(1+\delta)^{-1}\lambda R\,(a + br^{-2})\,v -$$
$$-\delta\,(1+\delta)^{-1}\tau\,(L - \lambda^2)\,\{[\lambda v_0 r^{-2} v - (\sigma + \tau)\,(L - \lambda^2)\,\psi]\,([(\sigma+\tau)^2 + v_0^2\,r^{-3}]^{-1}\}$$
$$[L - \lambda^2 - (1+\delta)^{-1}\sigma R]\,v = 2\,(1+\delta)^{-1}\lambda\,Ra\psi + \delta\,(1+\delta)^{-1}\tau\,(\sigma+\tau)^{-1}Lv - \tag{1.4}$$
$$-\delta\,(1+\delta)^{-1}\tau\lambda\,[\lambda\,(\sigma+\tau)\,v + v_0 r^{-1}(L - \lambda^2)\,\psi]\,[(\sigma+\tau)^2 + v_0^2\,r^{-3}]^{-1}$$
$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}, \qquad \psi(1) = \psi\left(\frac{R_2}{R_1}\right) = \frac{d\psi(1)}{dr} = \frac{d\psi(R_2/R_1)}{dr} =$$
$$= v(1) = v\left(\frac{R_2}{R_1}\right) = 0$$

Characteristic equation of the problem of stability of (1.2) has the form

$$F(\sigma, \lambda, \delta, \tau, R, R_2/R_1, \Omega_2/\Omega_1) = 0 \tag{1.5}$$

When $\delta = 0$, Eqs. (1.4) and (1.5) correspond to the problem of stability of the Couette flow of a Newtonian fluid. As usual [3] we assume that $\sigma = 0$, i.e. that the 'neutral' perturbation constitutes, actually, a secondary flow. In this case the critical value of R corresponding to the boundary of the region of instability will be the smallest positive root of Eq.

$$F(0, \lambda, \delta, \tau, R, R_2/R_1, \Omega_2/\Omega_1) = 0$$

Assuming that our investigation is of approximate character, we may postulate that the clearance between the cylinders is small and that the outer cylinder does not rotate [3 and 5]. Let us, in addition, assume that $\delta \ll 1$, i.e. that the antisymmetric component of the stress tensor is small, compared with the symmetric one. Then if $\tau \ll 1$, we can neglect the terms of (1.4) containing δ^2, $\delta\tau$ and τ^2 to obtain the following relations for u_1 and v_1:

$$(D^2 - k^2)^2 u_1 = 2\,(1-\delta)k^2 R' v_1, \qquad [D^2 - k^2\,(1+\delta)]\,v_1 = 2aR'u_1 \tag{1.6}$$
$$u_1 = v_1 = du_1/d\xi = 0 \qquad \text{when } \xi = 0,\ \xi = 1$$
$$\xi = (r-1)/\varepsilon, \qquad \varepsilon = (R_2 - R_1)/R_1 \ll 1, \qquad k = \lambda\varepsilon, \qquad R' = R\varepsilon^2$$
$$u_1 = u/\varepsilon = \psi k/\varepsilon^2$$
$$v_1 = v/\varepsilon, \qquad D(\ldots) = d(\ldots)/d\xi, \qquad a < 0$$

Eliminating u_1 from (1.6) we have

$$-(D^2 - k^2)^2 [D^2 - k^2 (1 + \delta)] v_1 = k^2 T (1 - \delta) v_1 \tag{1.7}$$

$$v_1 = 0, \quad [D^2 - k^2 (1 + \delta)] v_1 = 0, \quad D [D^2 - k^2 (1 + \delta)] v_1 = 0 \quad \text{when } \xi = 0,1$$

$$T = -2 \, aR'^2 = -2a (R_2 / R_1 - 1)^4 R^2 \tag{1.8}$$

Here T is the Taylor parameter and its lowest value defines the criterion of stability.

Putting $\delta = 0$ in (1.7) we obtain the well-known equation for a Newtonian fluid

$$-(D^2 - k^2) v_1 = T^0 k v_1, \qquad T^0 = -2a (R_2 / R_1 - 1)^4 R^2$$

$$v_1 = 0, \quad (D^2 - k^2) v_1 = 0, \quad D(D^2 - k^2) v_1 = 0 \quad \text{when } \xi = 0; 1 \tag{1.9}$$

which can also be obtained from (1.4) under the assumptions that $\tau \gg 1$ and $\delta \ll 1$.

In the following we shall denote v_1 and u_1 by v and u, respectively.

2. Following [4] we shall now reduce (1.6) to the integral operator form

$$v = \mu \, G_1 G_2 \, v, \qquad \mu = k^2 (1 - \delta) \, T \tag{2.1}$$

where G_j are Green's integral operators defined by

$$G_j f = \int_0^1 K_j (\xi, \xi_0) \, f(\xi_0) \, d\xi, \qquad j = 1, 2 \tag{2.2}$$

Here K_1 and K_2 are Green's functions for ordinary differential operators $[-D^2 + k^2(1 + \delta)]$ and $(D^2 - k^2)^2$ appearing in (1.6). We have

$$[D^2 - k^2 (1 + \delta)] f = \rho_0' \frac{d}{d\xi} \rho_1' \frac{d}{d\xi} \rho_2' f, \qquad \rho_0 = \rho_2 = e^{k\xi}$$

$$(D^2 - k^2)^2 f = \rho_0 \frac{d}{d\xi} \rho_1 \frac{d}{d\xi} \rho_2 \rho_0 \frac{d}{d\xi} \rho_1 \frac{d}{d\xi} \rho_2 f, \qquad \rho_1 = e^{-2k\xi}$$

$$\rho_0' = \rho_2' = e^{k(1+\delta/2)\xi}, \qquad \rho_1' = e^{-2k(1+\delta/2)\xi}$$

Functions ρ are positive, continuous and infinitely and continuously differentiable, therefore G_j are integral oscillatory operators [6 and 7].

The product $G_1 G_2$ is again an oscillatory operator, hence the spectrum of the problem (2.1) consists of a sequence of simple, positive eigenvalues [7]

$$0 < \mu_1 (k) < \ldots < \mu_n (k) \to \infty \tag{2.3}$$

It is easy to see that the operators G_j are linear, symmetric and completely continuous over a Hilbert space H° with the following scalar product:

$$(\varphi, \psi)_{H^\circ} = \int_0^1 \varphi(\xi) \, \psi(\xi) \, d\xi$$

In this case G_j will be analytic functions of the parameter k and all eigenvalues μ_n will also be analytic in k [4]. Relations (2.1) imply that a sequence of positive values T_n exists, for which neutral perturbations are possible and, moreover, that they are analytic functions of k (with exception of $k = 0$ and ∞).

Let us now assume that $0 < T_1 (k) < \ldots < T_n (k) \to \infty$. Here $T_1 (k)$ defines the boundary of the domain of stability. Let us denote by T^* the minimum value of $T_1 (k)$. Obviously $T^* > 0$. We shall show that T^* can be achieved at some $k > 0$. Since $T_1 (k)$ is analytic in k, it will be sufficient to show that $T_1 (0) = T_1 (\infty) = \infty$. We know that $\mu_1 (k) > 0$ for any k, therefore $T_1 (k) = \mu_1 (k)/k^2 \to \infty$ as $k \to 0$. To find the value of $T_1 (k)$ when $k \to \infty$, we shall multiply (1.6) by u and v respectively and integrate the result with respect to ξ from zero to one. Simple calculations will then yield the estimate $T_1 (k) > k^4$ from which it follows that $T_1 (k) \to \infty$ as $k \to \infty$.

3. Neglecting the terms containing δ^2, we can rewrite (1.7) as

$$-(D^2 - h^2)^3 v - 2\delta h^2 (D^2 - h^2)^2 v = \mu v, \qquad h^2 = k^2 (1 + \delta) \tag{3.1}$$

$$v = 0, \quad (D^2 - h^2) v = 0, \quad D (D^2 - h^2) v = 0 \quad \text{for } \xi = 0; 1$$

Let M be the set of functions $\{v\}$ which are fourfold continuously differentiable and which satisfy the boundary conditions (3.1). Let the Hilbert space H supplement the set on the norm generated by the scalar product

$$(v, f)_H = \int_0^1 (DvDf + h^2 vf)\, d\xi = -\int_0^1 [(D^2 - h^2)\, v]\, f\, d\xi \tag{3.2}$$

We shall define the operator A on H as follows:

$$(Av, f)_H = \int_0^1 \{[(D^2 - h^2)^4 + 2\delta h^2 (D^2 - h^2)^3]\, v\}\, f\, d\xi, \qquad f \in H \tag{3.3}$$

The system (3.1) is equivalent to the operator equation

$$Av = \mu v \tag{3.4}$$

We easily see that the operator A is self-adjoint on H. Putting $k = h$, we can write (1.9) in the operator form

$$A^\circ v = \mu^\circ v, \qquad \mu^\circ = h^2 T^\circ \tag{3.5}$$

where A° is given by

$$(A^\circ v, f)_H = \int_0^1 [(D^2 - h^2)^4 v]\, f\, d\xi, \qquad f \in H \tag{3.6}$$

The operator A° is also self-adjoint on H. Let us write A in the form $A = A^\circ + \delta A^1$, where A^1 is given by

$$(A^1 v, f)_H = 2\int_0^1 [(D^2 - h^2)^3 v]\, f\, d\xi \tag{3.7}$$

Since A and A° are self-adjoint and their eigenvalues are simple, we can expand the eigenvalues of A into a series [8]

$$\mu_1 = \mu_1^\circ + \delta\mu_1^1 + \ldots, \quad \mu_1^1 = (A^1 v_1^\circ, v_1^\circ)_H \tag{3.8}$$

where $v_1^\circ(h, \xi)$ is an eigenfunction corresponding to μ_1° – the first eigenvalue of the operator A° and μ_1 is the first eigenvalue of A.

Neglecting the squares of v_1° we obtain from (1.9), (2.2), (3.5) and (3.3).

$$T_1(k) = (1 + 2\delta) T^\circ_1(h) \tag{3.9}$$

which yields the approximate values of $T_1(k)$ provided that the solution of the problem of stability of the Couette flow of a Newtonian fluid is known.

Relation (3.9) shows that the Couette flow of the Grad fluid is more stable than that of a Newtonian fluid.

We also observe that (1.7) will be formally equivalent to the equations of the problem of stability of the Couette flow of a viscoplastic fluid [5] if we put in (1.7)

$$\delta = f_0 = \frac{1}{2}\left\{\left(\frac{2B}{S} - 1\right)^{-1} + \left[\frac{2B}{S(1+\varepsilon)^2} - 1\right]\right\}, \qquad B = \frac{S}{2} + \frac{\varepsilon}{2} + 1, \qquad S = \frac{\tau_0}{\Omega_1 \eta}$$

where τ_0 is the yield point and η is the dynamic coefficient of Newtonian viscosity.

Since (1.7) holds for $\delta \ll 1$, computational results of [5] can be utilised in determining the stability of the Couette flow of the Grad fluid for small f_0.

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